

An Impossibility Result for Reconstruction in a Degree-Corrected Planted-Partition Model

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Abstract

We consider a Degree-Corrected Planted-Partition model: a random graph on n nodes with two equal-sized clusters. The model parameters are two constants $a, b > 0$ and an i.i.d. sequence of weights $(\phi_u)_{u=1}^n$, with finite second moment $\Phi^{(2)}$. Vertices u and v are joined by an edge with probability $\frac{\phi_u \phi_v}{n} a$ when they are in the same class and with probability $\frac{\phi_u \phi_v}{n} b$ otherwise.

We prove that it is information-theoretically impossible to estimate the spins in a way positively correlated with the true community structure when $(a - b)^2 \Phi^{(2)} \leq 2(a + b)$.

A by-product of our proof is a precise coupling-result for local-neighbourhoods in Degree-Corrected Planted-Partition models, which could be of independent interest.

1 Introduction

It is well-known that many networks exhibit a community structure. Think about groups of friends, web pages discussing related topics, or people speaking the same language (for instance, the Belgium population could be roughly divided into people speaking either Flemish or French). Finding those communities helps us understand and exploit general networks.

Instead of looking directly at real networks, we experiment first with models for networks with communities. One of the most elementary models is the Planted-Partition Model (PPM) [14]: a random graph on n vertices partitioned into two equal-sized clusters such that vertices within the same cluster are connected with probability p and between the two communities with probability q . Note that the PPM is a special case of the Stochastic Block Model (SBM). The question is now: given an instance of the PPM, can we retrieve the community-membership of its vertices?

Most real networks are sparse and a thorough analysis of the sparse regime in the PPM - i.e., $p = \frac{a}{n}$ and $q = \frac{b}{n}$ for some constants $a, b > 0$ - will therefore lead to a better understanding of networks.

When the difference between a and b is small, the graph might not even contain enough information to distinguish between the two clusters. In [9] it was first conjectured that a detectability phase-transition exists in the PPM: detection would be possible if and only if $\left(\frac{a-b}{2}\right)^2 > \frac{a+b}{2}$. The negative-side of this conjecture has been confirmed in [25]. The positive side has been recently confirmed in [20] and [24] using sophisticated (but still running in polynomial time) algorithms designed for this particular problem.

In this paper we study an extension of the PPM: a Degree-Corrected Planted-Partition Model (DC-PPM), a special case of the Degree-Corrected Stochastic Block Model (DC-SBM) in [16]. Because, although the PPM is a useful model due to its

analytical tractability, it fails to accurately describe networks with a wide variety in their degree-sequences (because nodes in the same cluster are stochastically indistinguishable). Indeed, real degree distributions follow often, but not always, a power-law [1]. Compare this to fitting a straight line on intrinsically curved data, which is doomed to miss important information.

The DC-PPM is defined as follows: It is a random graph on n vertices partitioned into two asymptotically equal-sized clusters by giving each vertex v a spin $\sigma(v)$ drawn uniformly from $\{+, -\}$. The vertices have i.i.d. weights $\{\phi_u\}_{u=1}^n$ governed by some law ν with support in $W \subset [\phi_{\min}, \phi_{\max}]$, where $0 < \phi_{\min} \leq \phi_{\max} < \infty$ are constants. We denote the second moment of the weights by $\Phi^{(2)}$. An edge is drawn between nodes u and v with probability $\frac{\phi_u \phi_v}{n} a$ when u and v have the same spin and with probability $\frac{\phi_u \phi_v}{n} b$ otherwise. The model parameters a and b are constant.

In the underlying paper we extend results in [25] to the degree-corrected setting. More specifically, we prove that when $(a - b)^2 \Phi^{(2)} \leq 2(a + b)$, it is information-theoretically impossible to estimate the spins in a way positively correlated with the true community structure.

In a follow-up paper [13], we show that above the threshold (i.e., $(a - b)^2 \Phi^{(2)} > 2(a + b)$), reconstruction is possible based on the second eigenvector of the so-called non-backtracking matrix. This is an extension of the results in [3] for the *ordinary* Stochastic Block Model.

We remark that there is an interpretation of the threshold in terms of eigenvalues of the conditional expectation of A . Indeed, if A denotes the adjacency matrix and f_1 and f_2 are the vectors defined for $u \in V$ by $\psi_1(u) = \frac{1}{\sqrt{2}} \phi_u$ and $\psi_2(u) = \frac{1}{\sqrt{2}} \sigma_u \phi_u$, then

$$\mathbb{E}[A|\phi_1, \dots, \phi_n] = \frac{a+b}{n} \psi_1 \psi_1^* + \frac{a-b}{n} \psi_2 \psi_2^* - a \frac{1}{n} \text{diag}\{\phi_u^2\}.$$

So that, in probability,

$$\mathbb{E}[A|\phi_1, \dots, \phi_n] \psi_1 = \left(\frac{a+b}{2} \frac{1}{n} \sum_{u=1}^n \phi_u^2 \right) \psi_1 + \left(\frac{a-b}{2} \frac{1}{n} \sum_{u=1}^n \sigma_u \phi_u^2 \right) \psi_2 + \mathcal{O}(1) \rightarrow \frac{a+b}{2} \Phi^{(2)} \psi_1,$$

and

$$\mathbb{E}[A|\phi_1, \dots, \phi_n] \psi_2 = \left(\frac{a-b}{2} \frac{1}{n} \sum_{u=1}^n \phi_u^2 \right) \psi_2 + \left(\frac{a+b}{2} \frac{1}{n} \sum_{u=1}^n \sigma_u \phi_u^2 \right) \psi_1 + \mathcal{O}(1) \rightarrow \frac{a-b}{2} \Phi^{(2)} \psi_2,$$

by the law of large numbers. Now, the condition $(a - b)^2 \Phi^{(2)} \leq 2(a + b)$ is equivalent to $\left(\frac{a-b}{2} \Phi^{(2)} \right)^2 \leq \frac{a+b}{2} \Phi^{(2)}$.

1.1 Our results

It is a well-known fact that in the sparse graph, $\Theta(n)$ vertices are isolated whose type thus cannot be recovered by any algorithm. Therefore, the best we can ask for is that our reconstruction is positively correlated with the true partition:

Definition 1.1. *Let G be an observation of the degree-corrected planted partition-model, with true communities $\{\sigma_u\}_{u=1}^n$. Further, let $\{\hat{\sigma}_u\}_{u=1}^n$ be a reconstruction of the communities, based on the observation G , such that $|\{u : \hat{\sigma}_u = +\}| = \frac{n}{2}$. Then, we say that $\{\hat{\sigma}_u\}_{u=1}^n$ is positively correlated with the true partition $\{\sigma_u\}_{u=1}^n$ if there exists $\delta > 0$ such that*

$$\frac{1}{n} \sum_{u=1}^n 1_{\{\sigma_u = \hat{\sigma}_u\}} \geq \frac{1}{2} + \delta,$$

with high probability.

Our main result is as follows:

Theorem 1.2. *Let G be an observation of the degree-corrected planted partition-model with $(a-b)^2\Phi^2 \leq 2(a+b)$. Then, no reconstruction $\{\hat{\sigma}_u\}_{u=1}^n$ based on G is positively correlated with $\{\sigma_u\}_{u=1}^n$.*

As mentioned above, this theorem is an extension of the result in [25]. Their strategy, that we shall follow, invokes a connection with the tree reconstruction problem (see for instance [23]): deducing the sign of the root based on all the spins at some distance $R \rightarrow \infty$ from the root. Indeed, we shall see that the R -neighbourhood of a vertex looks like the following random $+/-$ labelled tree, that we denote by T^{Poi} :

We begin with a single particle, the root o , having spin $\sigma_o \in \{+, -\}$ and weight $\phi_o \in W \subset [\phi_{\min}, \phi_{\max}]$ (which we often take random). The root is replaced in generation 1 by $\text{Poi}\left(\frac{a}{2}\Phi^{(1)}\phi_o\right)$ particles of spin σ_o and $\text{Poi}\left(\frac{b}{2}\Phi^{(1)}\phi_o\right)$ particles of spin $-\sigma_o$. Further, the weights of those particles are i.i.d. distributed following law ν^* , the size-biased version of ν , defined for $x \in [\phi_{\min}, \phi_{\max}]$ by

$$\nu^*(x) = \frac{1}{\Phi^{(1)}} \int_{\phi_{\min}}^x y d\nu(y). \quad (1.1)$$

For generation $t \geq 1$, a particle with spin σ and weight ϕ^* is replaced in the next generation by $\text{Poi}\left(\frac{a}{2}\Phi^{(1)}\phi^*\right)$ particles with the same spin and $\text{Poi}\left(\frac{b}{2}\Phi^{(1)}\phi^*\right)$ particles of the opposite sign. Again, the weights of the particles in generation $t+1$ follow in an i.i.d. fashion the law ν^* . The offspring-size of an individual is thus a **Poisson-mixture**.

1.2 General proof idea

We first note that reconstruction is senseless when $\frac{a+b}{2}\Phi^{(2)} \leq 1$, because in this regime there is no giant component¹. Note further that $\frac{a+b}{2}\Phi^2 \leq 1$ already implies $(a-b)^2\Phi^2 \leq 2(a+b)$.

To prove that detection is not possible when $\frac{a+b}{2}\Phi^{(2)} > 1$ and $(a-b)^2\Phi^{(2)} \leq 2(a+b)$, we show in Theorem 4.2 that for uniformly chosen vertices u and v ,

$$\mathbb{P}(\sigma_u = + | \sigma_v, G) \xrightarrow{\mathbb{P}} \frac{1}{2}, \quad (1.2)$$

as $n \rightarrow \infty$. I.e., it is already impossible to decide the sign of two random vertices, which is an easier problem than reconstructing the group-membership of all vertices (made precise in Lemma 4.3). To establish (1.2), we condition on the boundary spins of an R neighbourhood around u , where R tends to infinity: it should make reconstruction easier. But, as we shall see, long-range correlations in this model are weak (Lemma 4.1). Hence, we can leave out the conditioning on the spin of v , so that we are precisely in the setting of a tree-reconstruction problem, see Section 2. In fact, we shall prove (Theorem 2.4) that reconstruction of the sign of the root in a T^{Poi} tree based on the spins at depth R (where $R \rightarrow \infty$), is impossible when $(a-b)^2\Phi^2 \leq 2(a+b)$.

1.3 Outline and differences with *ordinary* Planted-Partition model

Due to the presence of weights, the offspring in the branching process is governed by a Poisson-mixture. Section 2 deals with these type of branching processes. The main

¹Indeed, the main result in [2] concerns the existence, size and uniqueness of the giant component. In particular, in the setting considered here, a giant component emerges if and only if $\frac{a+b}{2}\Phi^2 > 1$. We shall henceforth assume a giant component to emerge.

theorem (i.e., Theorem 2.4) deals with a reconstruction problem on sequences of trees rather than a single random tree as in Theorem 4.1 in [25].

In Section 3 we establish a coupling between the local neighbourhood and T^{Poi} . This result is different from the coupling in [2], because we need the weights in the graph and their counterparts in the branching process to be *exactly* the same.

Finally, in Section 4 we show that long-range interactions are weak. The proof of Lemma 4.1 is based on an idea in the proof of Lemma 4.7 in [25]. Note however that (besides the presence of weights) the statement of our Lemma 4.1 is slightly stronger than Lemma 4.7 in [25], see below for details.

1.4 Background

Without the degree correction (i.e., $\phi_1 = \dots = \phi_n = 1$), the authors of [9] were the first to conjecture a phase-transition for the ordinary planted partition model based on ideas from statistical physics: Clustering positively correlated with the true spins is possible if $(a - b)^2 > 2(a + b)$ and impossible if $(a - b)^2 < 2(a + b)$. They conjectured further that using the so-called belief propagation algorithm would establish the positive part. In [17] 'spectral redemption conjecture' was made: detection using the second eigenvalue of the so called non-backtracking matrix would also establish the positive part.

The work [6] showed that positively correlated results exist in the sparse case, however not applying all the way down to the threshold. The remainder of the positive part in was established [20] by using a matrix counting the number of self-avoiding paths in the graph. The work [24] establishes independently of [20] the positive part of the conjecture in [9]. Further, the authors of [24] show impossibility in [25]. In fact they show a bit more, namely that for $(a - b)^2 \leq 2(a + b)$ reconstructions are never positively correlated. We shall here extend their results for the DC-PPM by relying on similar techniques.

Recovering the planted partition (without degree-corrections) often coincides with finding the minimum bisection in the same graph. That is, finding a partition of the graph such that the number of edges between separated components (the *bisection width*) is small. This problem is NP-hard [10].

Graph bisection on random graphs has been studied intensively. For instance, [4] studies the collection of labelled simple random graphs that have $2n$ nodes, node-degree d at least 3 and bisection width $o(n^{1 - \lfloor (d+1)/2 \rfloor})$. For these graphs the minimum bisection is much smaller than the average bisection. The main result is a polynomial-time algorithm based on the maxflow-mincut theorem, that finds exactly the minimum bisection for almost all graphs.

Another example is given in [10]. There, the authors consider the uniform distribution on the set of all graphs that have a small cut containing at most a fraction $1/2 - \epsilon$ of the total number of edges for some fixed $\epsilon > 0$. Those authors show that, if in the planted partition model $p > q$ are fixed (for $n \rightarrow \infty$), then the underlying community structure coincides with the minimum bisection and it can be retrieved in polynomial time. This result is improved in [15].

In [21] the case of non-constant p and q is analysed. A spectral algorithm is presented that recovers the communities with probability $1 - \delta$ if $p - q > c \left(\sqrt{p \frac{\log(n/\delta)}{n}} \right)$. Here c is a sufficiently large constant.

Positive results of spectral clustering in the DC-SBM have been obtained by various authors. The work [8] introduces a reconstruction algorithm based on the matrix that is obtained by dividing each element of the adjacency matrix by the geometric mean of its row and column degrees.

The extended planted-partition is studied in [7]. In that model, an edge is present between u and v with probability $(1_{\{\sigma_u = \sigma_v\}} a + 1_{\{\sigma_u \neq \sigma_v\}} b) \cdot (\phi_u \phi_v) / (\bar{\phi} n)$, where $\bar{\phi} =$

$\sum_{u=1}^n \phi_u$, the average weight. The main result is a polynomial time algorithm that outputs a partitioning that differs from the planted clusters on no more than $n \log(\bar{\phi})/\bar{\phi}^{0.98}$ nodes. This recovery succeeds only under certain conditions: the minimum weight should be a fraction of the average weight and the degree of each vertex is $o(n)$.

The article [18] gives an algorithm based on the adjacency matrix of a graph together with performance guarantees. The average degree should be at least of order $\log n$. However, since the spectrum of the adjacency matrix is dominated by the top eigenvalues [5], the algorithm does a poor job when the degree-sequence is very irregular.

The authors of the underlying paper propose in [12] an algorithm that recovers consistently the block-membership of all but a vanishing fraction of nodes, even when the lowest degree is of order $\log n$. It outperforms algorithms based on the adjacency matrix in case of heterogeneous degree-sequences.

2 Broadcasting on the branching process

Here we repeat without changes the definition of a Markov broadcasting process on trees given in [25]. Let \mathcal{T} be an infinite tree with root ρ . Given a number $0 \leq \epsilon < 1$, define a random labelling $\tau \in \{\pm\}^{\mathcal{T}}$. First, draw τ_ρ uniformly in $\{\pm\}$. Then, conditionally independently given τ_ρ , take every child u of ρ and set $\tau_u = \tau_\rho$ with probability $1 - \epsilon$ and $\tau_u = -\tau_\rho$ otherwise. Continue this construction recursively to obtain a labelling τ for which every vertex, independently, has probability $1 - \epsilon$ of having the same label as its parent.

Suppose that the labels $\tau_{\partial\mathcal{T}_m}$ at depth m in the tree are known (here, $\tau_W = \{\tau_i : i \in W\}$). The paper [11] gives precise conditions as to when reconstruction of the root label is feasible using the optimal reconstruction strategy (maximum likelihood), i.e., deciding according to the sign of $\mathbb{E}[\tau_\rho | \tau_{\partial\mathcal{T}_m}]$. Interestingly, this is completely decided by the branching number of \mathcal{T} .

Definition 2.1. *The branching number of a tree \mathcal{T} , denoted by $\text{Br}(\mathcal{T})$, is defined as follows:*

- If \mathcal{T} is finite, then $\text{Br}(\mathcal{T}) = 0$;
- If \mathcal{T} is infinite, then $\text{Br}(\mathcal{T}) = \sup\{\lambda \geq 1 : \inf_{\Pi} \sum_{v \in \Pi} \lambda^{-|v|} > 0\}$, where the infimum is taken over all cutsets Π .

Theorem 1.1 in [11] reads tailored to our needs:

Theorem 2.2. *(Theorem 1.1 in [11]) Consider the problem of reconstructing τ_ρ from the spins $\tau_{\partial\mathcal{T}_m}$ at the m th level of \mathcal{T} . Define Δ_m as the difference between the probability of correct and incorrect reconstruction given the information at level m :*

$$\Delta_m := |\mathbb{P}(\tau_\rho = + | \tau_{\partial\mathcal{T}_m}) - \mathbb{P}(\tau_\rho = - | \tau_{\partial\mathcal{T}_m})|.$$

*If $\text{Br}(\mathcal{T})(1 - 2\epsilon)^2 > 1$ then $\lim_{m \rightarrow \infty} \mathbb{E}[\Delta_m] > 0$.
If, however, $\text{Br}(\mathcal{T})(1 - 2\epsilon)^2 < 1$ then $\lim_{m \rightarrow \infty} \mathbb{E}[\Delta_m] = 0$.*

Note that in this theorem the tree is fixed, compared to the setting in this paper where the multi-type branching process T^{Poi} defined in Section 1.1 is considered. But, it can be easily seen that the spins on a fixed instance \mathcal{T} of T^{Poi} are distributed according to the above broadcasting process with error probability $\epsilon = \frac{b}{a+b}$ ². We thus need to calculate the branching number of a typical instance \mathcal{T} :

²Indeed, instances of the tree *when ignoring spins* are generated according to a Galton-Watson process where the number of offspring of a particle is an independent copy of $\text{Poi}\left(\frac{a+b}{2}\Phi^{(1)}\phi^*\right)$, with ϕ^* governed by ν^* . We obtain the spins by giving particles, independently, the same spin as its parent with probability $\frac{a}{a+b}$ and the opposite sign with probability $\frac{b}{a+b}$.

Proposition 2.3. Assume that $\frac{a+b}{2}\Phi^{(2)} > 1$. Consider the multi-type branching process T^{Poi} , where the root has spin drawn uniformly from $\{+, -\}$ and weight governed by ν . Then, given the event that the branching process does not go extinct, $\text{Br}(T^{\text{Poi}}) \leq \frac{a+b}{2}\Phi^{(2)}$ almost surely.

Proof. Denote the multi-type branching process by T . Assume w.l.o.g. that the root has $D \geq 1$ children denoted as $1, \dots, D$. Denote by T_u^* the subtree of all particles with common ancestor u . We observe that $\text{Br}(T) < c$ if and only if $\text{Br}(T_u^*) < c$ for all u .

Now, conditional on the spin of the root, $(T_u^*)_{u=1}^D$ are i.i.d. copies of T^{Poi} with weight governed by the biased law ν^* . The latter is a Galton-Watson process with offspring mean $\frac{a+b}{2}\Phi^{(2)} > 1$. Hence Proposition 6.4 in [19] entails that $\text{Br}(T_u^*) = \frac{a+b}{2}\Phi^{(2)}$ a.s. \square

Note that it can in fact be easily proved that $\text{Br}(T^{\text{Poi}}) = \frac{a+b}{2}\Phi^{(2)}$ almost surely.

We conclude with the main theorem of this section. Note that we assume that $\frac{a+b}{2}\Phi^{(2)} > 1$, so that the branching process does not die out with non-zero probability. Remark further that the theorem is a bit more precise than Theorem 4.1 in [25] (which deals with *unweighed* Poisson trees), in the sense that we need to *re-sample* the tree for each n . Indeed, in the coupling result Theorem 3.1 below we re-sample for each n .

Theorem 2.4. Assume that $\frac{a+b}{2}\Phi^{(2)} > 1$. Let $\{T^n\}_{n=1}^\infty$ be a collection of i.i.d. copies of T^{Poi} . Denote for each tree T^n its spins by τ^n . Further, let R be an unbounded non-decreasing function. Assume that $(a-b)^2\Phi^2 < 2(a+b)$, then

$$\mathbb{P}\left(\tau_\rho^n = + | T_{R(n)}^n, \tau_{\partial T_{R(n)}^n}^n\right) \xrightarrow{\mathbb{P}} \frac{1}{2},$$

as $n \rightarrow \infty$.

Proof. We begin by describing the above broadcasting process on random trees more precisely. By the triple $(\Omega, \Sigma, \mathbb{P})$ we denote the underlying probability-space of the following stochastic process: Let T be the branching process T^{Poi} with root ρ where we ignore all types on it (we denote the collection of its realizations by Ω' , and we let Σ' be a sigma-algebra on it). We define a new random labelling $\tau \in \{\pm\}^T$ on every instance of T by running the Markov broadcast process. I.e., τ_ρ is uniformly drawn from $\{\pm\}$ and each child has the same spin as its parent with probability $\frac{a}{a+b}$.

Let, for each $n \in \mathbb{N}$, (T^n, τ^n) be an independent copy of (T, τ) . Formally, the random variable T^n is thus a mapping from Ω to Ω' : it therefore makes sense to define the pull-back measure $\mathbb{P}_T : \Sigma' \rightarrow [0, 1]$ for $B \in \Sigma'$ by $\mathbb{P}_T(B) = \mathbb{P}((T^1)^\leftarrow(B))$.

With this notation,

$$\begin{aligned} \mathbb{E}\left[\left|\mathbb{P}\left(\tau_\rho^n = + | T_{R(n)}^n, \tau_{\partial T_{R(n)}^n}^n\right) - \frac{1}{2}\right|\right] &= \int_{\Omega'} \mathbb{E}\left[\left|\mathbb{P}\left(\tau_\rho^n = + | T_{R(n)}^n, \tau_{\partial T_{R(n)}^n}^n\right) - \frac{1}{2}\right| \middle| T^n = \mathcal{T}\right] d\mathbb{P}_T(\mathcal{T}) \\ &= \int_{\Omega'} \mathbb{E}\left[\left|\mathbb{P}\left(\tau_\rho = + | T_{R(n)}, \tau_{\partial T_{R(n)}}\right) - \frac{1}{2}\right| \middle| T = \mathcal{T}\right] d\mathbb{P}_T(\mathcal{T}) \\ &= \int_{\Omega'} \mathbb{E}\left[\left|\mathbb{P}\left(\tau_\rho = + | T = \mathcal{T}, \tau_{\partial T_{R(n)}}\right) - \frac{1}{2}\right|\right] d\mathbb{P}_T(\mathcal{T}). \end{aligned}$$

Since $\text{Br}(T) \leq \frac{a+b}{2}\Phi^2$ almost surely, $\text{Br}(T)(1 - 2\epsilon) < 1$ almost surely. Consequently,

$$f_n(\mathcal{T}) := \mathbb{E}\left[\left|\mathbb{P}\left(\tau_\rho = + | T = \mathcal{T}, \tau_{\partial T_{R(n)}}\right) - \frac{1}{2}\right|\right] \rightarrow 0$$

Thus, a particle gives birth to $\sum_{u=1}^{\text{Poi}\left(\frac{a+b}{2}\Phi^{(1)}\phi^*\right)} 1_{\frac{a}{a+b}} \stackrel{d}{=} \text{Poi}\left(\frac{a}{2}\Phi^{(1)}\phi^*\right)$ particles of the same sign, and $\sum_{u=1}^{\text{Poi}\left(\frac{a+b}{2}\Phi^{(1)}\phi^*\right)} \left(1 - 1_{\frac{a}{a+b}}\right) \stackrel{d}{=} \text{Poi}\left(\frac{b}{2}\Phi^{(1)}\phi^*\right)$ particles of the opposite sign. Those numbers are seen to be independent.

for almost every realization \mathcal{T} of T . Because $f_n(\cdot) \leq 1/2$, it is the immediate consequence of Lebesgue's dominated convergence theorem that

$$\mathbb{P}\left(\tau_\rho^n = +|T_{R(n)}^n, \tau_{\partial T_{R(n)}^n}^n\right) \rightarrow \frac{1}{2},$$

in $L^1(\Omega, \Sigma, \mathbb{P})$ as $n \rightarrow \infty$.

Finally, it is a well-known fact that convergence in L^1 implies convergence in probability. \square

3 Coupling of local neighbourhood

This section has as its objective to establish a coupling between the local neighbourhood of an arbitrary fixed vertex in the DC-PPM and T^{Poi} . The main result is the following theorem, where we let T , τ , and ψ be random instances of T^{Poi} , its spins and its weights, respectively.

Theorem 3.1. *Let $R(n) = C \log(n)$, with $C < \frac{1 - \log(4/e)}{3 \log(2\kappa_{\max})}$. Let ρ be a uniformly picked vertex in $V(G)$, where for each n , $G = G(n)$ is an instance of the DC-PPM. Let for each n , (T^n, τ^n, ψ^n) be an independent copy of (T, τ, ψ) , then*

$$\mathbb{P}\left((G_{R(n)}(\rho), \sigma_{G_R}, \phi_{G_R}) = (T_{R(n)}^n, \tau_{T_R}^n, \psi_{T_R}^n)\right) = 1 - n^{-\frac{1}{2} \log(4/e)}.$$

We defer its proof to the end of this section. It uses an alternative description of the branching process in Section 1.1.

3.1 Alternative description of branching process

We obtain an alternative description of the graph by considering a particle u with spin σ_u and weight ϕ_u to be of type $x_u = \phi_u \sigma_u \in S = -W \cup W$. We denote the law of x_u by μ , i.e., for $A \subset S$, $\mu(A) = \int_A \frac{1}{2} d\nu(|x|)$. Two distinct vertices u and v are then joined by an edge with probability $\frac{\kappa(x_u, x_v)}{n}$, where $\kappa : S \times S \rightarrow \mathbb{R}$ is defined for $(x, y) \in S \times S$ by

$$\kappa(x, y) = |xy| (1_{\{xy > 0\}} a + 1_{\{xy < 0\}} b). \quad (3.1)$$

Analogously, we obtain the following equivalent description of the branching process: We begin with a single particle o of type x_o governed by μ , giving birth to $\text{Poi}(\lambda_{x_o}(S))$ children, where for $x \in S$, and $A \subset S$,

$$\lambda_x(A) = \int_A \kappa(x, y) d\mu(y). \quad (3.2)$$

Conditional on x_o the children have i.i.d. types governed by $\mu_{x_o}^*$ ³, where for $x \in S$, and $A \subset S$,

$$\mu_x^*(A) = \frac{\lambda_x(A)}{\lambda_x(S)} = \int_A \left(\frac{a}{a+b} 1_{xy > 0} + \frac{b}{a+b} 1_{xy < 0} \right) |y| \frac{d\nu(|y|)}{\Phi(1)}. \quad (3.3)$$

For generation $t \geq 1$, all particles give birth independently in the following way: A particle with type x^* is replaced in the next generation by $\text{Poi}(\lambda_{x^*}(S))$ children, again with i.i.d. types governed by $\mu_{x^*}^*$.

³Note that if y has law μ_x^* , then for any $A \subset W$, $\mathbb{P}(\text{sign}(y) = \text{sign}(x), |y| \in A) = \frac{a}{a+b} \int_A y \frac{dy}{\Phi(1)} = \mathbb{P}(\text{sign}(y) = \text{sign}(x)) \mathbb{P}(|y| \in A)$. Hence, we can identify $\text{sign}(y)$ with the particle's spin and $|y|$ with its independent weight.

In [2] it is shown that local neighbourhoods of the graph are described by the above branching process, if we *ignore* the types. (To be precise: the equivalent description used in [2] is that a particle of type x gives birth to $\text{Poi}(\lambda_x(A))$ children with type in A , for any $A \subset S$. Those numbers are independent for different sets A and different particles.)

The coupling-technique in [2] uses a discretization of κ as an intermediate step, thereby losing some information: types in the tree deviate slightly from their counterparts in the graph. We shall therefore use another coupling method, presented below, so that **the types in graph and branching process are exactly the same**.

3.2 Coupling

We use the following exploration process: At time $m = 0$, choose a vertex ρ uniformly in $V(G)$, where G is an instance of the DC-PPM. Initially, it is the only active vertex: $\mathcal{A}(0) = \{\rho\}$. All other vertices are neutral at start: $\mathcal{U}(0) = V(G) \setminus \{\rho\}$. No vertex has been explored yet: $\mathcal{E}(0) = \emptyset$. At each time $m \geq 0$ we arbitrarily pick an active vertex u in $\mathcal{A}(m)$ that has shortest distance to ρ , and explore all its edges in $\{uv : v \in \mathcal{U}(m)\}$: if $uv \in E(G)$ for $v \in \mathcal{U}(m)$, then we set v active in step $m + 1$, otherwise it remains neutral. At the end of step m , we designate u to be explored. Thus,

$$\mathcal{E}(m+1) = \mathcal{E}(m) \cup \{u\},$$

$$\mathcal{A}(m+1) = (\mathcal{A}(m) \setminus \{u\}) \cup (\mathcal{N}(u) \cap \mathcal{U}(m)),$$

and,

$$\mathcal{U}(m+1) = \mathcal{U}(m) \setminus \mathcal{N}(u).$$

Our aim in this section is to show that the exploration process and the branching process are equal upto depth $R(n)$ (defined in Theorem 3.1) with probability tending to one for large n . We do this in two steps:

Firstly, we establish that the i.i.d. vertices in $\mathcal{U}(m)$ follow a law $\mu^{(m)}$ such that

$$\left\| \mu^{(m)} - \mu \right\|_{\text{TV}} = \mathcal{O}\left(\frac{m}{n}\right).$$

This is the content of the following:

Lemma 3.2. *Let $1, \dots, m$ be the vertices in $\mathcal{E}(m)$, with types $X_1 = x_1, \dots, X_m = x_m$. Then, the vertices in $\mathcal{U}(m)$ are i.i.d. with law $\mu^{(m)} = \mu_{x_1, \dots, x_m}^{(m)}$, where*

$$d\mu^{(m)}(\cdot) = \frac{g(\cdot)d\mu(\cdot)}{\int_S g(z)d\mu(z)}, \quad (3.4)$$

with

$$g(\cdot) = \prod_{i=1}^m \left(1 - \frac{\kappa(x_i, \cdot)}{n}\right). \quad (3.5)$$

Further, in the regime $m = o(n)$, there exists N_m such that for all (x_1, \dots, x_m) :

$$\left\| \mu_{x_1, \dots, x_m}^{(m)} - \mu \right\|_{\text{TV}} \leq 2\kappa_{\max} \frac{m}{n},$$

if $n \geq N_m$.

Secondly, if u has type $X = x \in S$, then its D neighbours in $\mathcal{U}(m)$ (i.e., those vertices that will be added to $\mathcal{A}(m+1)$) are i.i.d. with law $\mu_x^{*(m+1)}$, which is $\mathcal{O}\left(\frac{m}{n}\right)$ away from μ_x^* in total variation distance. Further, we can approximate the number of neighbours D by $\text{Poi}(\lambda_x(S))$ with error $\mathcal{O}\left(\frac{|\mathcal{U}(m)|-n}{n}\right) + \mathcal{O}\left(\frac{m}{n}\right)$:

Lemma 3.3. Assume u has type $X = x$. Let D be the number of neighbours u has in $\mathcal{U}(m)$. Then, the types of those neighbours are i.i.d. with law $\mu_x^{*(m)}$, where

$$d\mu_x^{*(m)}(\cdot) = \frac{\kappa(x, \cdot) d\mu^{(m)}(\cdot)}{\int_S \kappa(x, y) d\mu^{(m)}(y)}. \quad (3.6)$$

Recall N_m from Lemma 3.2: if $n \geq N_m$ then

$$\left\| \mu_x^{*(m)} - \mu_x^* \right\|_{TV} \leq 4 \frac{\kappa_{max}^3}{\kappa_{min}^2} \frac{m}{n}. \quad (3.7)$$

Further,

$$\|D - \text{Poi}(\lambda_x(S))\|_{TV} \leq \kappa_{max} \frac{n - |\mathcal{U}(m)|}{n} + 3\kappa_{max}^2 \frac{m}{n}. \quad (3.8)$$

To establish the desired coupling, let us give names to all good events:

$$A_{r+1} = \{\forall u \in \partial G_r : D_u = \widehat{D}_u\},$$

$$B_{r+1} = \{\forall u \in \partial G_r, v \in \{1, \dots, D_u\} : U_{uv} = \widehat{U}_{uv}\},$$

$$C_r = \{|\partial G_s| \leq g(s) = 2^s M^s \log(n) \ \forall s \leq r\},$$

where, for $u \in \partial G_r$ (we identify $\partial G_r = \{1, \dots, |\partial G_r|\}$),

- $D_u = |\mathcal{N}(u) \cap \mathcal{U}(|G_{r-1}| + u - 1)|$;

and where, conditional that u has type $X_u = x_u$,

- $\widehat{D}_u = \text{Poi}(\lambda_{x_u}(S))$;
- moreover, for $v \in \{1, \dots, D_u\}$:
 - U_{uv} denotes the type of child v of vertex u ;
 - \widehat{U}_{uv} is a random variable with law $\mu_{x_u}^*$.

The types attached to siblings are independent conditional on their parents type.

With the above lemma's established, the event

$$E_r = \bigcap_{s=1}^r \{A_s \cap B_s \cap C_s\}$$

happens indeed w.h.p.:

Lemma 3.4. For any integer $r \leq R(n)$,

$$\mathbb{P}(E_{r+1}|E_r) \geq 1 - n^{3C \log(2\kappa_{max})-1} - n^{-\log(4/e)},$$

for large enough n .

The events A_r , B_r and C_r alone do not contain enough information to completely reconstruct the neighbourhood of ρ : vertices in ∂G_r might be merged among each other, or it is possible that they share a child in V_r . But, those events, are rare. Indeed, let K_r be the event that no vertex in G_r has more than one neighbour outside G_r and that there are no edges in ∂G_r . Then, we have the following:

Lemma 3.5. Let $r \leq R$, then

$$\mathbb{P}(K_r|C_R) \geq 1 - n^{3C \log(2\kappa_{max})-1},$$

for large enough n .

Proof of Lemma 3.2. Consider vertex $v \in \mathcal{U}(m)$ with type Y . We show first that, conditional on $v \notin \mathcal{N}(1, \dots, m)$ and $X_1 = x_1, \dots, X_m = x_m$, Y has law $\mu_{x_1, \dots, x_m}^{(m)}$. To this end we shall calculate for $y \in S$,

$$\begin{aligned} & \mathbb{P}(Y \leq y | v \notin \mathcal{N}(1, \dots, m), X_1 = x_1, \dots, X_m = x_m) \\ &= \frac{\mathbb{P}(Y \leq y) \mathbb{P}(v \notin \mathcal{N}(1, \dots, m) | Y \leq y, X_1 = x_1, \dots, X_m = x_m)}{\mathbb{P}(v \notin \mathcal{N}(1, \dots, m) | X_1 = x_1, \dots, X_m = x_m)}, \end{aligned} \quad (3.9)$$

since $\mathbb{P}(Y \leq y | X_1 = x_1, \dots, X_m = x_m) = \mathbb{P}(Y \leq y)$. Recall (3.5) and observe that

$$g(\cdot) = \mathbb{P}(v \notin \mathcal{N}(1, \dots, m) | Y = \cdot, X_1 = x_1, \dots, X_m = x_m).$$

Hence, the denominator in (3.9) is just

$$\mathbb{P}(v \notin \mathcal{N}(1, \dots, m) | X_1 = x_1, \dots, X_m = x_m) = \int_S g(z) d\mu(z). \quad (3.10)$$

Evaluating the numerator yields,

$$\begin{aligned} & \mathbb{P}(Y \leq y) \mathbb{P}(v \notin \mathcal{N}(1, \dots, m) | Y \leq y, X_1 = x_1, \dots, X_m = x_m) \\ &= \mathbb{P}(Y \leq y) \int_{-\phi_{\min}}^y g(z) dp(z) = \int_{-\phi_{\min}}^y g(z) d\mu(z), \end{aligned} \quad (3.11)$$

where, we defined for $z \leq y$, $p(z) = \mathbb{P}(Y \leq z | Y \leq y, X_1 = x_1, \dots, X_m = x_m)$. By combining (3.10) and (3.11) we establish (3.4), i.e., Y has distribution $\mu^{(m)}$.

To see that $\mu^{(m)}$ indeed approximates μ , observe that

$$\left(1 - \frac{\kappa_{\max}}{n}\right)^m \leq \frac{g(\cdot)}{\int_S g(z) d\mu(z)} \leq \frac{1}{\left(1 - \frac{\kappa_{\max}}{n}\right)^m}.$$

We use Taylor's theorem to appropriately bound both sides for large n :

$$\left(1 - \frac{\kappa_{\max}}{n}\right)^m = 1 - \kappa_{\max} \frac{m}{n} + \mathcal{O}\left(\frac{\kappa_{\max}^2}{2} \left(\frac{m}{n}\right)^2\right),$$

where $\left|\mathcal{O}\left(\frac{\kappa_{\max}^2}{2} \left(\frac{m}{n}\right)^2\right)\right| \leq \frac{\kappa_{\max}^2}{2} \left(\frac{m}{n}\right)^2$ and,

$$\frac{1}{\left(1 - \frac{\kappa_{\max}}{n}\right)^m} = 1 + \kappa_{\max} \frac{m}{n} + \mathcal{O}\left(4\kappa_{\max}^2 \left(\frac{m}{n}\right)^2\right),$$

where $\left|\mathcal{O}\left(4\kappa_{\max}^2 \left(\frac{m}{n}\right)^2\right)\right| \leq 4\kappa_{\max}^2 \left(\frac{m}{n}\right)^2$. Consequently, there exists N_m such that

$$\left\| \mu^{(m)} - \mu \right\|_{\text{TV}} \leq \int_S \left| \frac{g(y)}{\int_S g(z) d\mu(z)} - 1 \right| d\mu(y) \leq 2\kappa_{\max} \frac{m}{n},$$

for all $n \geq N_m$. \square

Proof of Lemma 3.3. Put $n_m = |\mathcal{U}(m)|$ and let Y_1, \dots, Y_D denote the types of the neighbours of u .

Let f be an arbitrary measurable function. The first claim follows if we prove that

$$\mathbb{E} \left[e^{-\sum_{j=1}^D f(Y_j)} \middle| D = d \right] = \left(\int_S e^{-f(y)} d\mu_x^{*(m)}(y) \right)^d \quad (3.12)$$

Now,

$$\begin{aligned} & \mathbb{E} \left[e^{-\sum_{j=1}^D f(Y_j)} 1_{D=d} \right] \\ &= \sum_{F \subset [n_m], |F|=d} \mathbb{E} \left[e^{-\sum_{j \in F} f(Y_j)} \middle| F \right] \left(1 - \frac{1}{n} \int_S \kappa(x, y) d\mu^{(m)}(y) \right)^{n_m-d} \left(\frac{1}{n} \int_S \kappa(x, y) d\mu^{(m)}(y) \right)^d, \end{aligned}$$

where, conditioning on F means that $\mathcal{N}(u) \cap \mathcal{U}(m) = F$. We have,

$$\mathbb{P}(D = d) = \binom{n_m}{d} \left(1 - \frac{1}{n} \int_S \kappa(x, y) d\mu^{(m)}(y)\right)^{n_m-d} \left(\frac{1}{n} \int_S \kappa(x, y) d\mu^{(m)}(y)\right)^d.$$

Hence,

$$\mathbb{E} \left[e^{-\sum_{j=1}^D f(Y_j)} \middle| D = d \right] = \frac{1}{\binom{n_m}{d}} \sum_{F \subset [n_m], |F|=d} \mathbb{E} \left[e^{-\sum_{j \in F} f(Y_j)} \middle| F \right].$$

Conditional on $F \subset [n_m]$, the types $(Y_j)_{j \in F}$ are i.i.d., thus

$$\mathbb{E} \left[e^{-\sum_{j \in F} f(Y_j)} \middle| F \right] = \left(\frac{\int_S e^{-f(y)} \frac{\kappa(x, y)}{n} d\mu^{(m)}(y)}{\int_S \frac{\kappa(x, y)}{n} d\mu^{(m)}(y)} \right)^d,$$

which combined with (3.6) gives (3.12), our first claim.

Further, an explicit calculation yields

$$\int_S |d\mu_x^{*(m)} - d\mu_x^*| \leq 4 \frac{\kappa_{\max}^3}{\kappa_{\min}^2} \frac{m}{n},$$

establishing (3.7).

For the last claim, observe that $D = \text{Bin}(n_m, p)$, where $p = \frac{1}{n} \int_S \kappa(x, y) d\mu^{(m)}(y)$. Hence,

$$\|\text{Bin}(n_m, p) - \text{Poi}(n_m p)\|_{\text{TV}} \leq \sum_{i=1}^{n_m} p^2 \leq \frac{\kappa_{\max}^2}{n}.$$

Standard bounds for Poisson random variables entail the existence of a constant $C_{\text{Poi}} \geq 1$ such that $\|\text{Poi}(\mu) - \text{Poi}(\lambda)\|_{\text{TV}} \leq C_{\text{Poi}} |\mu - \lambda|$. Consequently,

$$\begin{aligned} \frac{1}{C_{\text{Poi}}} \|\text{Poi}(n_m p) - \text{Poi}(\lambda_x(S))\|_{\text{TV}} &\leq |n_m - n|p + \left| \int_S \kappa(x, y) d\mu^{(m)}(y) - \int_S \kappa(x, y) d\mu(y) \right| \\ &\leq \kappa_{\max} \frac{|n_m - n|}{n} + \kappa_{\max} \int_S |d\mu^{(m)}(y) - d\mu(y)| \\ &\leq \kappa_{\max} \frac{|n_m - n|}{n} + 2\kappa_{\max}^2 \frac{m}{n}. \end{aligned}$$

Thus,

$$\|\text{Bin}(n_m, p) - \text{Poi}(\lambda_x(S))\|_{\text{TV}} \leq C_{\text{Poi}} \left(\kappa_{\max} \frac{|n_m - n|}{n} + 3\kappa_{\max}^2 \frac{m}{n} \right).$$

□

Proof of Lemma 3.4. Write $n_r = |\partial G_r|$. We have

$$\mathbb{P}(E_{r+1}|E_r) \geq \mathbb{P}(B_{r+1}|E_r) - \mathbb{P}(\neg A_{r+1}|E_r) - \mathbb{P}(\neg C_{r+1}|E_r).$$

Now,

$$\mathbb{P}(B_{r+1}|E_r, n_r) \geq 1 - \sum_{u=1}^{n_r} \mathbb{P} \left(\neg B_{r+1}^{(u)} \middle| \bigcap_{v=1}^{u-1} B_{r+1}^{(v)}, E_r \right), \quad (3.13)$$

where $B_{r+1}^{(u)} = \{\forall w \in \{1, \dots, D_u\} : U_{uw} = \widehat{U}_{uw}\}$. Denote the already explored vertices by $1, \dots, m$ (where $m = |G_{r-1}| + u - 1$) and their types as X_1, \dots, X_m . Conditional on those types, the vertices in $\mathcal{U}(m)$ are i.i.d. with distribution $\mu^{(m)}$. Hence:

$$\begin{aligned} \mathbb{P} \left(B_{r+1}^{(u)} \middle| \bigcap_{v=1}^{u-1} B_{r+1}^{(v)}, E_r, n_r, X_1, \dots, X_m \right) &= \mathbb{P} \left(B_{r+1}^{(u)} \middle| X_1, \dots, X_m \right) \\ &\geq \mathbb{P} \left(B_{r+1}^{(u)} \middle| D_u \leq (1 + \log(n))\kappa_{\max}, X_1, \dots, X_m \right) \mathbb{P}(D_u \leq (1 + \log(n))\kappa_{\max} | X_1, \dots, X_m). \end{aligned} \quad (3.14)$$

Now, $D_u \stackrel{d}{\leq} \text{Bin}(n, \frac{\kappa_{\max}}{n})$, regardless of X_1, \dots, X_m . Consequently, due to a multiplicative Chernoff bound,

$$\mathbb{P}(D_j \leq (1 + \log(n))\kappa_{\max} | X_1, \dots, X_m) \geq 1 - \frac{1}{n^3}, \quad (3.15)$$

for large enough n . Lemma 3.3 entails

$$\mathbb{P}\left(B_{r+1}^{(u)} | D_u \leq (1 + \log(n))\kappa_{\max}, X_1, \dots, X_m\right) \geq 1 - c_1 \frac{m \log(n)}{n}. \quad (3.16)$$

Then, (3.14) - (3.16) together give

$$\mathbb{P}\left(B_{r+1}^{(u)} \middle| \bigcap_{v=1}^{u-1} B_{r+1}^{(v)}, E_r, X_1, \dots, X_m\right) \geq 1 - c_2 \frac{m \log(n)}{n}.$$

Now, since $m \leq |G_r| \leq rg(r)$ and $n_r \leq g(r)$, (3.13) gives

$$\mathbb{P}(B_{r+1} | E_r) \geq 1 - \frac{rg^2(r) \log^2(n)}{n}.$$

We take a similar approach to quantify

$$\mathbb{P}(A_{r+1} | E_r, n_r) \geq 1 - \sum_{u=1}^{n_r} \mathbb{P}\left(\neg A_{r+1}^{(u)} \middle| \bigcap_{v=1}^{u-1} A_{r+1}^{(v)}, E_r, n_r\right), \quad (3.17)$$

where, $A_{r+1}^{(u)} = \{D_u = \hat{D}_u, D_u \leq (1 + \log(n))\kappa_{\max}\}$. Now,

$$\begin{aligned} & \mathbb{P}\left(A_{r+1}^{(u)} \middle| \bigcap_{v=1}^{u-1} A_{r+1}^{(v)}, E_r\right) \geq \\ & \mathbb{P}\left(D_u = \hat{D}_u \middle| \bigcap_{v=1}^{u-1} A_{r+1}^{(v)}, E_r\right) - \mathbb{P}\left(D_u > (1 + \log(n)) \middle| \bigcap_{v=1}^{u-1} A_{r+1}^{(v)}, E_r\right) \\ & \geq 1 - \frac{rg(r) \log^2(n)}{n}, \end{aligned} \quad (3.18)$$

due to Lemma 3.3, since $n - |\mathcal{U}(m)| \leq |G_r| + (u-1)(1 + \log(n))\kappa_{\max}$. Thus, (3.17) gives

$$\mathbb{P}(A_{r+1} | E_r) \geq 1 - \frac{rg^2(r) \log^2(n)}{n}.$$

We finish by establishing the growth condition (i.e., C_{r+1}): On C_r , $|\partial G_r| \leq (2\kappa_{\max})^r \log(n)$, thus

$$|\partial G_{r+1}| \leq Z := \text{Bin}((2\kappa_{\max})^r \log(n)n, \frac{\kappa_{\max}}{n}).$$

Hence,

$$\begin{aligned} \mathbb{P}(\neg C_{r+1} | E_r) &= \mathbb{P}(|\partial G_{r+1}| > 2^{r+1} \kappa_{\max}^{r+1} \log(n) | E_r) \\ &\leq \mathbb{P}(Z > 2\mathbb{E}[Z]) \\ &\leq \left(\frac{e}{4}\right)^{\mathbb{E}[Z]}, \end{aligned}$$

by a multiplicative version of Chernoff's bound. Now, $\mathbb{E}[Z] = 2^r \kappa_{\max}^{r+1} \log(n)$, hence,

$$\mathbb{P}(C_{r+1} | E_r) \geq 1 - \frac{1}{n^{\log(4/e)}}.$$

□

Proof of Lemma 3.5. Fix $u, v \in \partial G_r$. The probability of having an edge between u and v is smaller than $\frac{\kappa_{\max}}{n}$. For any $w \in V(G \setminus G_r)$, the probability that (u, w) and (v, w) both appear is smaller than $\frac{\kappa_{\max}^2}{n^2}$. Now, Lemma 3.4 implies that

$$|G_r| \leq g(R)R(n) = n^{C \log(2\kappa_{\max})} C \log^2(n) = o(n^\beta),$$

for all $\beta > C \log(2\kappa_{\max})$. Hence, the result follows from a union bound over all triples u, v, w . \square

Proof of Theorem 3.1. We have

$$\mathbb{P}(\neg E_R) \leq \sum_{r=1}^R \mathbb{P}(\neg E_r | E_{r-1}) \leq R(n) \left(n^{3C \log(2\kappa_{\max})-1} + n^{-\log(4/e)} \right). \quad (3.19)$$

Similarly,

$$\mathbb{P}\left(\bigcap_{s=1}^R K_s | E_R\right) \leq R(n) n^{3C \log(2\kappa_{\max})-1}.$$

Hence, due to the choice of C ,

$$\mathbb{P}_n\left(\bigcap_{s=1}^R K_s, E_R\right) \geq 1 - n^{-\frac{1}{2} \log(4/e)}.$$

\square

4 No long-range correlation in DC-PPM

In this section we establish (1.2). To this end, we first condition on both the spins of $\partial G_{R(n)}$ and all weights in G . Lemma 4.1 below shows that we then can remove the conditioning on σ_v and the graph structure outside the R -neighbourhood (including the weights):

$$\text{Var}(\sigma_u | \sigma_{\partial G_R}, \sigma_v, G, \phi) = \text{Var}(\sigma_u | \sigma_{\partial G_R}, G_R, \phi_{G_R}) + o_n(1). \quad (4.1)$$

We established in the previous section that a neighbourhood in G looks like a T^{Poi} tree with a Markov broadcasting process on it. Hence, the right-hand side of (4.1) converges to 1 in probability, establishing (1.2). We show in Theorem 4.3 below that this contradicts the existence of a reconstructed bisection that is positively correlated with the true type-assignment.

We begin by preparing an auxiliary lemma to prove (1.2), it establishes that long-range interactions are sufficiently weak. Its proof is inspired by Lemma 4.7 in [25]. However (besides the additional complication of weights) the result stated here is stronger in the sense that the $o_n(1)$ terms converge uniformly to 0 and that "conditioning on G " may now be replaced with "conditioning on $G_{A \cup B}$ ".

Lemma 4.1. *Let G be an instance of the DC-PPM. Let u be an uniformly picked vertex in $V(G)$. Let $A = A(G)$, $B = B(G)$, $C = C(G) \subset V$ be a (random) partition of $V(G)$, with $u \in A$, such that B separates A and C in G . If $|A \cup B| \leq n^{1/8}$ for asymptotically almost every realization of G , then*

$$|\mathbb{P}(\sigma_u = + | \sigma_{B \cup C}, G, \phi) - \mathbb{P}(\sigma_u = + | \sigma_B, G_{A \cup B}, \phi_{A \cup B})| = o_n(1), \quad (4.2)$$

with probability $1 - o_n(1)$. Moreover, the $o_n(1)$ terms in (4.2) converge uniformly to 0.

Proof. For a fixed graph g , spin-configuration τ and degree-configuration ψ , we make a factorization of $\mathbb{P}(G = g, \sigma = \tau | \phi = \psi)$ into parts depending on A, B and C . We claim that the part that measures the interaction between A and C can be neglected with respect to the other parts. Put

$$\Psi_{uv}(g, \tau, \psi) = \begin{cases} a \frac{\psi_u \psi_v}{n} & \text{if } (u, v) \in E(g) \text{ and } \tau_u = \tau_v \\ b \frac{\psi_u \psi_v}{n} & \text{if } (u, v) \in E(g) \text{ and } \tau_u \neq \tau_v \\ 1 - a \frac{\psi_u \psi_v}{n} & \text{if } (u, v) \notin E(g) \text{ and } \tau_u = \tau_v \\ 1 - b \frac{\psi_u \psi_v}{n} & \text{if } (u, v) \notin E(g) \text{ and } \tau_u \neq \tau_v. \end{cases}$$

We define for arbitrary sets $U_1, U_2 \subset V$,

$$Q_{U_1, U_2} = Q_{U_1, U_2}(g, \tau, \psi) = Q_{U_1, U_2}(g_{U_1 \cup U_2}, \tau_{U_1 \cup U_2}, \psi_{U_1 \cup U_2}) = \prod_{u \in U_1, v \in U_2} \Psi_{uv}(g, \tau, \psi),$$

where the subscript indicates restriction of the corresponding quantities to $U_1 \cup U_2$. Then, we have,

$$\mathbb{P}(G = g | \sigma = \tau, \phi = \psi) = Q_{A \cup B, A \cup B} Q_{B \cup C, C} Q_{A, C}. \quad (4.3)$$

We begin by demonstrating that $Q_{A, C}$ is asymptotically independent of τ : Write,

$$Q_{A, C}(g, \tau, \psi) = \prod_{u \in A, v \in C: \tau_u = \tau_v} \left(1 - a \frac{\psi_u \psi_v}{n}\right) \prod_{u \in A, v \in C: \tau_u \neq \tau_v} \left(1 - b \frac{\psi_u \psi_v}{n}\right),$$

since A and C are separated by B (there are thus no edges between A and C). The first product may be rewritten as,

$$\begin{aligned} \prod_{u \in A, v \in C: \tau_u = \tau_v} \left(1 - a \frac{\psi_u \psi_v}{n}\right) &= \exp \left(\sum_{u \in A, v \in C: \tau_u = \tau_v} \log \left(1 - a \frac{\psi_u \psi_v}{n}\right) \right) \\ &= \exp \left(\sum_{u \in A, v \in C: \tau_u = \tau_v} \left(-a \frac{\psi_u \psi_v}{n} + \mathcal{O}(1/n^2) \right) \right) \\ &= \exp \left(-\frac{a}{n} \sum_{u \in A, v \in C: \tau_u = \tau_v} \psi_u \psi_v \right) \exp(\mathcal{O}(n_A n_C / n^2)). \end{aligned}$$

Now, the sum $\sum_{u \in A, v \in C: \tau_u = \tau_v} \psi_u \psi_v$ tends to $\frac{\|A\| \|C\|}{2}$, if $(\tau, \psi) \in \Omega(n)$, where

$$\|U\| = \sum_{u \in U} \psi_u, \quad (U \subset V),$$

and where,

$$\Omega(n) = \{(\tau', \psi') : S_C(\psi', \tau') \leq n^{3/4}\}, \quad (4.4)$$

with, for $U \subset V$,

$$S_U(\psi, \tau) = \sum_{u \in U} \psi_u \tau_u.$$

To prove that $\sum_{u \in A, v \in C: \tau_u = \tau_v} \psi_u \psi_v$ indeed converges to $\frac{\|A\| \|C\|}{2}$, we introduce the following quantities:

$$|U^\pm|(\psi, \tau) = \sum_{u \in U: \tau_u = \pm} \psi_u, \quad (U \subset V),$$

to write,

$$\|A\| \|C\| + S_A S_C = 2(|A^+ \|C^+| + |A^- \|C^-|).$$

We use the latter observation to rewrite

$$\sum_{u \in A, v \in C: \tau_u = \tau_v} \psi_u \psi_v = |A^+||C^+| + |A^-||C^-| = \frac{\|A\|\|C\| + S_A S_C}{2},$$

where $S_A S_C \leq n_A n^{3/4} = n^{\frac{7}{8}}$, for $(\tau, \psi) \in \Omega(n)$. As a consequence,

$$\begin{aligned} \prod_{u \in A, v \in C: \tau_u = \tau_v} \left(1 - a \frac{\psi_u \psi_v}{n}\right) &= \exp\left(\mathcal{O}\left(\frac{n^{7/8}}{n}\right)\right) \exp\left(\mathcal{O}\left(\frac{n_A n_C}{n^2}\right)\right) \exp\left(-\frac{a\|A\|\|C\|}{2n}\right) \\ &= (1 + o_n(1)) \exp\left(-\frac{a\|A\|\|C\|}{2n}\right), \end{aligned}$$

where the o_n term is uniform for all $(\tau, \psi) \in \Omega(n)$. We carry out a similar calculation for the other product. Together we obtain

$$Q_{A,C}(g, \tau, \psi) = (1 + o_n(1)) \exp\left(-\frac{a+b}{2} \frac{\|A\|\|C\|}{n}\right), \quad (4.5)$$

uniformly for all $(\tau, \psi) \in \Omega(n)$. This proves that $Q_{A,C}(g, \tau, \psi)$ is indeed essentially independent of τ for most pairs (τ, ψ) .

We use the above to prove that, for $u \in V$,

$$\begin{aligned} \mathbb{P}(\sigma_u = \tau_u | \sigma_{B \cup C} = \tau_{B \cup C}, G = g, \phi = \psi, (\phi, \sigma) \in \Omega(n)) \\ = (1 + o_n(1)) \mathbb{P}(\sigma_u = \tau_u | \sigma_B = \tau_B, G_{A \cup B} = g_{A \cup B}, \phi_{A \cup B} = \psi_{A \cup B}, (\phi, \sigma) \in \Omega(n)) + o_n(1). \end{aligned} \quad (4.6)$$

Fix $(\tau, \psi) \in \Omega(n)$. Then,

$$\begin{aligned} \mathbb{P}(G = g, \sigma = \tau | \phi = \psi, (\phi, \sigma) \in \Omega(n)) &= \frac{\mathbb{P}(G = g, \sigma = \tau, (\phi, \sigma) \in \Omega(n) | \phi = \psi)}{\mathbb{P}((\phi, \sigma) \in \Omega(n) | \phi = \psi)} \\ &= \frac{\mathbb{P}(G = g, \sigma = \tau, (\phi, \sigma) \in \Omega(n) | \phi = \psi)}{\mathbb{P}(\sigma = \tau | \phi = \psi)} \frac{\mathbb{P}(\sigma = \tau | \phi = \psi)}{\mathbb{P}((\phi, \sigma) \in \Omega(n) | \phi = \psi)} \\ &= \mathbb{P}(G = g | \sigma = \tau, \phi = \psi) \frac{2^{-n}}{\mathbb{P}((\phi, \sigma) \in \Omega(n) | \phi = \psi)} \\ &= \mathbb{P}(G = g | \sigma = \tau, \phi = \psi) f(\psi, n), \end{aligned} \quad (4.7)$$

for some function f . Hence, plugging (4.3) and (4.5) in (4.7),

$$\begin{aligned} \mathbb{P}(G = g, \sigma = \tau | \phi = \psi, (\phi, \sigma) \in \Omega(n)) \\ = Q_{A \cup B, A \cup B}(g, \tau, \psi) Q_{B \cup C, C}(g, \tau, \psi) (1 + o_n(1)) \exp\left(-\frac{a+b}{2} \frac{\|A\|\|C\|}{n}\right) f(\psi, n). \end{aligned} \quad (4.8)$$

Put, for $U \subset V$,

$$\Omega_U(n) = \Omega_U(\psi, \tau_U, n) = \{\tau' : \tau'_U = \tau_U, (\tau', \psi) \in \Omega(n)\},$$

then, invoking (4.8),

$$\begin{aligned}
\mathbb{P}(G = g, \sigma_U = \tau_U | \phi = \psi, (\phi, \sigma) \in \Omega(n)) &= \sum_{\tau' \in \Omega_U(n)} \mathbb{P}(G = g, \sigma = \tau' | \phi = \psi, (\phi, \sigma) \in \Omega(n)) \\
&= \sum_{\tau' \in \Omega_U(n)} Q_{A \cup B, A \cup B}(g, \tau', \psi) Q_{B \cup C, C}(g, \tau', \psi) \\
&\quad \cdot (1 + o_n(1)) \exp\left(-\frac{a+b}{2} \frac{\|A\| \|C\|}{n}\right) f(\psi, n) \\
&= (1 + o_n(1)) \exp\left(-\frac{a+b}{2} \frac{\|A\| \|C\|}{n}\right) f(\psi, n) \\
&\quad \cdot \sum_{\tau' \in \Omega_U(n)} Q_{A \cup B, A \cup B}(g, \tau', \psi) Q_{B \cup C, C}(g, \tau', \psi),
\end{aligned} \tag{4.9}$$

where we could interchange the order $o_n(1)$ term and the sum because the former holds *uniformly* for all $(\phi, \sigma) \in \Omega(n)$.

We apply (4.9) with $U = A$ and $U = A \cup B$, to rewrite the right hand side of

$$\mathbb{P}(\sigma_A = \tau_A | \sigma_B = \tau_B, G = g, \phi = \psi, (\phi, \sigma) \in \Omega(n)) = \frac{\mathbb{P}(G = g, \sigma_{A \cup B} = \tau_{A \cup B} | \phi = \psi, (\phi, \sigma) \in \Omega(n))}{\mathbb{P}(G = g, \sigma_B = \tau_B | \phi = \psi, (\phi, \sigma) \in \Omega(n))} \tag{4.10}$$

as

$$\begin{aligned}
&(1 + o_n(1)) \frac{\sum_{\tau' \in \Omega_{A \cup B}(n)} Q_{A \cup B, A \cup B}(g, \tau', \psi) Q_{B \cup C, C}(g, \tau', \psi)}{\sum_{\tau' \in \Omega_B(n)} Q_{A \cup B, A \cup B}(g, \tau', \psi) Q_{B \cup C, C}(g, \tau', \psi)} \\
&= (1 + o_n(1)) \frac{Q_{A \cup B, A \cup B}(g, \tau, \psi) \sum_{\tau' \in \Omega_{A \cup B}(n)} Q_{B \cup C, C}(g, \tau', \psi)}{\sum_{\tau''' \in \Omega_{B \cup C}(n)} Q_{A \cup B, A \cup B}(g, \tau''', \psi) \sum_{\tau'' \in \Omega_{A \cup B}(n)} Q_{B \cup C, C}(g, \tau'', \psi)},
\end{aligned}$$

where we used that $Q_{U_1, U_2}(\tau')$ depends on τ' only through $\tau'_{U_1 \cup U_2}$ to rewrite the numerator. Factorization of the denominator is justified as follows: For an arbitrary $\tau' \in \Omega_B(n)$, put $\tau'' = (\tau_{A \cup B}, \tau'_C) \in \Omega_{A \cup B}(n)$ and $\tau''' = (\tau'_A, \tau_{B \cup C}) \in \Omega_{B \cup C}(n)$. Then,

$$Q_{A \cup B, A \cup B}(g, \tau', \psi) Q_{B \cup C, C}(g, \tau', \psi) = Q_{A \cup B, A \cup B}(g, \tau''', \psi) Q_{B \cup C, C}(g, \tau'', \psi). \tag{4.11}$$

This proves that the double summation is at least as large as the single sum. Equality follows upon putting $\tau' = (\tau''', \tau_B, \tau'_C)$ for arbitrary $\tau'' \in \Omega_{A \cup B}(n)$ and $\tau''' \in \Omega_{B \cup C}(n)$: (4.11) is then again satisfied. Hence, (4.10) is equivalent to

$$\mathbb{P}(\sigma_A = \tau_A | \sigma_B = \tau_B, G = g, \phi = \psi, (\phi, \sigma) \in \Omega(n)) = (1 + o_n(1)) \frac{Q_{A \cup B, A \cup B}(g, \tau, \psi)}{\sum_{\tau''' \in \Omega_{B \cup C}(n)} Q_{A \cup B, A \cup B}(g, \tau''', \psi)}. \tag{4.12}$$

We shall rewrite the right hand side of (4.12) to obtain on the one hand:

$$\mathbb{P}(\sigma_u = \tau_u | \sigma_B = \tau_B, G = g, \phi = \psi, (\phi, \sigma) \in \Omega(n)) = (1 + o_n(1)) \widehat{F}(g_{A \cup B}, \tau_{u \cup B}, \psi_{A \cup B}), \tag{4.13}$$

for some function $\widehat{F}(\cdot) \leq 1$. And, on the other hand:

$$\begin{aligned}
&\mathbb{P}(\sigma_u = \tau_u | \sigma_B = \tau_B, G = g, \phi = \psi, (\phi, \sigma) \in \Omega(n)) \\
&= (1 + o_n(1)) \mathbb{P}(\sigma_u = \tau_u | \sigma_{B \cup C} = \tau_{B \cup C}, G = g, \phi = \psi, (\phi, \sigma) \in \Omega(n)).
\end{aligned} \tag{4.14}$$

To do so, note that

$$\sum_{\tau''' \in \Omega_{B \cup C}(n)} Q_{A \cup B, A \cup B}(g, \tau''', \psi) = \sum_{\tau_A'' \in \{\pm\}^A} Q_{A \cup B, A \cup B}(g_{A \cup B}, (\tau_A''', \tau_B), \psi_{A \cup B}).$$

Consequently,

$$\begin{aligned} \frac{Q_{A \cup B, A \cup B}(g, \tau, \psi)}{\sum_{\tau''' \in \Omega_{B \cup C}(n)} Q_{A \cup B, A \cup B}(g, \tau''', \psi)} &= \frac{Q_{A \cup B, A \cup B}(g_{A \cup B}, \tau_{A \cup B}, \psi_{A \cup B})}{\sum_{\tau_A''' \in \{\pm\}^A} Q_{A \cup B, A \cup B}(g_{A \cup B}, (\tau_A''', \tau_B), \psi_{A \cup B})} \\ &= F(g_{A \cup B}, \tau_{A \cup B}, \psi_{A \cup B}), \end{aligned}$$

for some function $F(\cdot) \leq 1$. Therefore, (4.12) is equivalent to

$$\mathbb{P}(\sigma_A = \tau_A | \sigma_B = \tau_B, G = g, \phi = \psi, (\phi, \sigma) \in \Omega(n)) = (1 + o_n(1)) F(g_{A \cup B}, \tau_{A \cup B}, \psi_{A \cup B}).$$

If we fix $u \in A$ and integrate over all possible values of $\tau_{A \setminus u}$ while keeping $\tau_{B \cup C}$ and ψ constant, we obtain (4.13) (since the above formula holds for any τ such that $(\tau, \psi) \in \Omega(n)$ and the latter condition depends only on the value of τ_C and ψ_C).

To establish (4.14), we multiply both denominator and enumerator of (4.12) by $Q_{B \cup C, C}(g, \tau, \psi)$:

$$\begin{aligned} &\mathbb{P}(\sigma_A = \tau_A | \sigma_B = \tau_B, G = g, \phi = \psi, (\phi, \sigma) \in \Omega(n)) \\ &= (1 + o_n(1)) \frac{Q_{A \cup B, A \cup B}(g, \tau, \psi) Q_{B \cup C, C}(g, \tau, \psi)}{\sum_{\tau' \in \Omega_{B \cup C}(n)} Q_{A \cup B, A \cup B}(g, \tau', \psi) Q_{B \cup C, C}(g, \tau', \psi)} \\ &= (1 + o_n(1)) \frac{\mathbb{P}(G = g, \sigma = \tau | \phi = \psi, (\phi, \sigma) \in \Omega(n))}{\mathbb{P}(G = g, \sigma_{B \cup C} = \tau_{B \cup C} | \phi = \psi, (\phi, \sigma) \in \Omega(n))} \\ &= (1 + o_n(1)) \mathbb{P}(\sigma_A = \tau_A | \sigma_{B \cup C} = \tau_{B \cup C}, G = g, \phi = \psi, (\phi, \sigma) \in \Omega(n)). \end{aligned}$$

Integrating again over $\tau_{A \setminus u}$ gives (4.14).

We use (4.13) to obtain

$$\begin{aligned} &\mathbb{P}(\sigma_u = \tau_u | \sigma_B = \tau_B, G_{A \cup B} = g_{A \cup B}, \phi_{A \cup B} = \psi_{A \cup B}, (\phi, \sigma) \in \Omega(n)) \\ &= \sum_{g_C, \psi_C} \mathbb{P}(\sigma_u = \tau_u | \sigma_B = \tau_B, G = (g_{A \cup B}, g_C), \phi = (\psi_{A \cup B}, \psi_C), (\phi, \sigma) \in \Omega(n)) \\ &\quad \cdot \mathbb{P}(G_C = g_C, \phi_C = \psi_C | \sigma_B = \tau_B, G_{A \cup B} = g_{A \cup B}, \phi_{A \cup B} = \psi_{A \cup B}, (\phi, \sigma) \in \Omega(n)) \\ &= \sum_{g_C, \psi_C} (1 + o_n(1)) \hat{F}(g_{A \cup B}, \tau_{u \cup B}, \psi_{A \cup B}) \\ &\quad \cdot \mathbb{P}(G_C = g_C, \phi_C = \psi_C | \sigma_B = \tau_B, G_{A \cup B} = g_{A \cup B}, \phi_{A \cup B} = \psi_{A \cup B}, (\phi, \sigma) \in \Omega(n)) \\ &= (1 + o_n(1)) \hat{F}(g_{A \cup B}, \tau_{u \cup B}, \psi_{A \cup B}) + o_n(1) \\ &= (1 + o_n(1)) \mathbb{P}(\sigma_u = \tau_u | \sigma_B = \tau_B, G = g, \phi = \psi, (\phi, \sigma) \in \Omega(n)) + o_n(1). \end{aligned} \tag{4.15}$$

Combining (4.14) and (4.15) gives

$$\begin{aligned} &\mathbb{P}(\sigma_u = \tau_u | \sigma_{B \cup C} = \tau_{B \cup C}, G = g, \phi = \psi, (\phi, \sigma) \in \Omega(n)) \\ &= (1 + o_n(1)) \mathbb{P}(\sigma_u = \tau_u | \sigma_B = \tau_B, G = g, \phi = \psi, (\phi, \sigma) \in \Omega(n)) \\ &= (1 + o_n(1)) \mathbb{P}(\sigma_u = \tau_u | \sigma_B = \tau_B, G_{A \cup B} = g_{A \cup B}, \phi_{A \cup B} = \psi_{A \cup B}, (\phi, \sigma) \in \Omega(n)), \end{aligned}$$

i.e., the claim (4.6).

Our last step consists in removing the condition $(\sigma, \phi) \in \Omega(n)$: Put $\epsilon(n) = 1 - \mathbb{P}((\sigma, \phi) \in \Omega(n))$ and note that $\lim_{n \rightarrow \infty} \epsilon(n) = 0$. Consider the random variable $\mathbb{P}((\phi, \sigma) \in \Omega(n) | \sigma_B, G_{A \cup B}, \phi_{A \cup B}) = \mathbb{E}[1_{(\phi, \sigma) \in \Omega(n)} | \sigma_B, G_{A \cup B}, \phi_{A \cup B}]$. It has expectation $1 - \epsilon(n)$, so that

$$\mathbb{P}\left(\mathbb{E}[1_{(\phi, \sigma) \in \Omega(n)} | \sigma_B, G_{A \cup B}, \phi_{A \cup B}] \geq 1 - \sqrt{\epsilon(n)}\right) \geq 1 - 2\sqrt{\epsilon(n)}. \tag{4.16}$$

Indeed, if contrary to our claim $f := \mathbb{E}[1_{(\phi, \sigma) \in \Omega(n)} | \sigma_B, G_{A \cup B}, \phi_{A \cup B}] \geq 1 - \sqrt{\epsilon(n)}$ with probability at most $1 - 2\sqrt{\epsilon(n)}$, then

$$\mathbb{E}[f] \leq 1 \cdot (1 - 2\sqrt{\epsilon(n)}) + (1 - \sqrt{\epsilon(n)}) \cdot 2\sqrt{\epsilon(n)} < 1 - \epsilon(n).$$

Similarly, for $B \cup C$,

$$\mathbb{P} \left(\mathbb{E} [1_{(\phi, \sigma) \in \Omega(n)} | \sigma_{B \cup C}, G, \phi] \geq 1 - \sqrt{\epsilon(n)} \right) \geq 1 - 2\sqrt{\epsilon(n)}. \quad (4.17)$$

Because $(1 + o_n(1)) \left(1 - \mathcal{O} \left(\sqrt{\epsilon(n)} \right) \right) = (1 + o_n(1))$, it follows that, with probability at least $1 - 4\sqrt{\epsilon(n)}$,

$$\begin{aligned} \mathbb{P}(\sigma_u = + | \sigma_B, G_{A \cup B}, \phi_{A \cup B}) &= \left(1 - \mathcal{O} \left(\sqrt{\epsilon(n)} \right) \right) \mathbb{P}(\sigma_u = + | \sigma_B, G_{A \cup B}, \phi_{A \cup B}, (\phi, \sigma) \in \Omega(n)) \\ &\quad + \mathcal{O} \left(\sqrt{\epsilon(n)} \right) \mathbb{P}(\sigma_u = + | \sigma_B, G_{A \cup B}, \phi_{A \cup B}, (\phi, \sigma) \notin \Omega(n)) \\ &= (1 + o_n(1)) \mathbb{P}(\sigma_u = + | \sigma_{B \cup C}, G, \phi, (\phi, \sigma) \in \Omega(n)) + o_n(1) \\ &= (1 + o_n(1)) \mathbb{P}(\sigma_u = + | \sigma_{B \cup C}, G, \phi) + o_n(1), \end{aligned}$$

where we used (4.16), (4.6) and (4.17) in the first, second, respectively last equality. \square

Theorem 4.2. Assume that $\frac{a+b}{2}\Phi^{(2)} \geq 1$ and $(a-b)^2\Phi^{(2)} \leq 2(a+b)$. Let G be an instance of the DC-PPM. Let u and v be uniformly chosen vertices in G . Then,

$$\mathbb{P}(\sigma_u = + | \sigma_v, G) \xrightarrow{\mathbb{P}} \frac{1}{2},$$

as $n \rightarrow \infty$.

Proof. Put $A = G_{R-1}$, $B = \partial G_R$ and $C = G \setminus G_R$. We use the monotonicity property of conditional variance:

$$1 \geq \text{Var}(\sigma_u | \sigma_v, G) \geq \text{Var}(\sigma_u | \sigma_{B \cup C}, G, \phi).$$

Now, by using the partition $A \cup B \cup C$ of $V(G)$ in Lemma 4.1, we have, since $G_R \leq n^{1/8}$ w.h.p.,

$$\mathbb{P}(\sigma_u = + | \sigma_{B \cup C}, G, \phi) \stackrel{w.h.p.}{=} \mathbb{P}(\sigma_u = + | \sigma_{\partial G_R}, G_R, \phi_{G_R}) + o_n(1).$$

Theorem 3.1 entails that the local neighbourhood is w.h.p. equal to T^{Poi} . Let T^n be an independent copy of T^{Poi} with root ρ , spins τ^n and weights ψ^n . Note that we stress the dependence on n , because the Poisson-tree is sampled again for each n .

$$\begin{aligned} \mathbb{P}(\sigma_u = + | \sigma_{\partial G_R}, G_R, \phi_{G_R}) + o_n(1) &\stackrel{w.h.p.}{=} \mathbb{P}(\tau_\rho^n = + | \tau_{\partial T_R^n}^n, T_R^n, \psi_{T_R^n}^n) + o_n(1) \\ &= \mathbb{P}(\tau_\rho^n = + | \tau_{\partial T_R^n}^n, T_R^n) + o_n(1), \end{aligned} \quad (4.18)$$

due to the coupling from Theorem 3.1. By Theorem 2.4, the right-hand side of (4.18) tends to $1/2$ in probability. Hence $\text{Var}(\sigma_u | \sigma_v, G)$ tends to 1 in probability. \square

Hence, if $\frac{a+b}{2}\Phi^{(2)} > 1$ and $(a-b)^2\Phi^{(2)} \leq 2(a+b)$, detection is not feasible:

Lemma 4.3. Assume that $\frac{a+b}{2}\Phi^{(2)} > 1$ and $(a-b)^2\Phi^{(2)} \leq 2(a+b)$. Let G be an observation of the degree-corrected planted partition-model, with true communities $\{\sigma_i\}_{i=1}^n$. Let $\{\hat{\sigma}_i\}_{i=1}^n$ be a reconstruction of the communities, based on the observation G , such that $|\{i : \hat{\sigma}_i = +\}| = \frac{n}{2}$. Assume that there exists $\delta > 0$ such that

$$f(n) := \frac{1}{n} \sum_{i=1}^n 1_{\{\sigma_i = \hat{\sigma}_i\}} \geq \frac{1}{2} + \delta,$$

with high probability. Then, $\mathbb{P}(\sigma_u = + | \sigma_v = +, G)$ does not converge in probability to $1/2$.

Proof. Assume for a contradiction that $\mathbb{P}(\sigma_u = + | \sigma_v = +, G) \xrightarrow{\mathbb{P}} \frac{1}{2}$. It suffices to analyse $\text{Var}(1_{\{\sigma_u = \sigma_v\}} | G)$. Indeed,

$$\mathbb{P}(\sigma_u = \sigma_v | G) = \frac{1/2 \cdot \mathbb{P}(\sigma_u = \sigma_v, G)}{1/2 \cdot \mathbb{P}(G)} = \frac{\mathbb{P}(\sigma_u = +, \sigma_v = +, G)}{\mathbb{P}(\sigma_v = +, G)} = \mathbb{P}(\sigma_u = + | \sigma_v = +, G),$$

so that

$$\begin{aligned} \text{Var}(1_{\{\sigma_u = \sigma_v\}} | G) &= \mathbb{E}[1_{\{\sigma_u = \sigma_v\}}^2 | G] - \mathbb{E}[1_{\{\sigma_u = \sigma_v\}} | G]^2 \\ &= \mathbb{P}(\sigma_u = \sigma_v | G) - \mathbb{P}(\sigma_u = \sigma_v | G)^2 \xrightarrow{\mathbb{P}} 1/4, \end{aligned}$$

as n tends to infinity. But, by the monotonicity of conditional variances,

$$\text{Var}(1_{\{\sigma_u = \sigma_v\}} | G) \leq \text{Var}(1_{\{\sigma_u = \sigma_v\}} | \hat{\sigma}_u = \hat{\sigma}_v),$$

for any estimator $(\hat{\sigma}_i)_{i=1}^n$ that is based on an observation of G . We shall show that this inequality is violated on the event that $\hat{\sigma}_u = \hat{\sigma}_v = +$.

Note that our reconstruction is an exact bisection, whereas the true community structure might be unevenly distributed. However, deviations are small (denote $n_{\pm} = |\{i : \sigma_i = \pm\}|$): The event $E = \{n_+ \in (\frac{n}{2} - n^{3/4}, \frac{n}{2} + n^{3/4})\}$ happens with probability larger than $1 - 2\exp(-n^{1/2})$.

Let $\epsilon > 0$ and condition on E . Assume that there are $(\frac{1}{2} + \epsilon)\frac{n}{2}$ vertices such that their true type and assigned type are both $+$. Then there are $\frac{n}{2} + \mathcal{O}(n^{3/4}) - (\frac{1}{2} + \epsilon)\frac{n}{2} = \frac{n}{4} - \epsilon\frac{n}{2} + \mathcal{O}(n^{3/4})$ vertices that have type $+$, but are assigned type $-$. Hence, $\frac{n}{2} - (\frac{n}{4} - \epsilon\frac{n}{2} + \mathcal{O}(n^{3/4})) = (\frac{1}{2} + \epsilon)\frac{n}{2} + \mathcal{O}(n^{3/4})$ vertices have true type and assigned type $-$. Thus,

$$\mathbb{P}\left(\sigma_u = \sigma_v = + \mid \hat{\sigma}_u = \hat{\sigma}_v = +, f(n) = \frac{1}{2} + \epsilon, E\right) = \frac{(\frac{1}{2} + \epsilon)\frac{n}{2}}{\frac{n}{2}} \cdot \frac{(\frac{1}{2} + \epsilon)\frac{n}{2} - 1}{\frac{n}{2} - 1} + \mathcal{O}(n^{-1/4}) \rightarrow \left(\frac{1}{2} + \epsilon\right)^2,$$

for large n . Similarly,

$$\mathbb{P}\left(\sigma_u = \sigma_v = - \mid \hat{\sigma}_u = \hat{\sigma}_v = +, f(n) = \frac{1}{2} + \epsilon, E\right) \rightarrow \left(\frac{1}{2} - \epsilon\right)^2,$$

for large n . Hence,

$$\mathbb{P}\left(\sigma_u = \sigma_v \mid \hat{\sigma}_u = \hat{\sigma}_v = +, f(n) = \frac{1}{2} + \epsilon, E\right) \rightarrow \frac{1}{2} + 2\epsilon^2,$$

uniformly for, say, all $\epsilon \geq \frac{\delta}{2}$. Consequently, there exists $N = N(\delta)$ such that for any $\epsilon \geq \frac{\delta}{2}$,

$$\mathbb{P}\left(\sigma_u = \sigma_v \mid \hat{\sigma}_u = \hat{\sigma}_v = +, f(n) = \frac{1}{2} + \epsilon, E\right) \geq \frac{1}{2} + \epsilon^2,$$

for all $n \geq N(\delta)$. Consequently,

$$\mathbb{P}(\sigma_u = \sigma_v | \hat{\sigma}_u = \hat{\sigma}_v = +) \geq \frac{1}{2} + \frac{\delta^2}{8},$$

for n large. Since $x \mapsto x - x^2$ is decreasing on $(1/2, 1)$, we have

$$\begin{aligned} \text{Var}(1_{\{\sigma_u = \sigma_v\}} | \hat{\sigma}_u = \hat{\sigma}_v = +) &= \mathbb{P}(1_{\{\sigma_u = \sigma_v\}} | \hat{\sigma}_u = \hat{\sigma}_v = +) - \mathbb{P}(1_{\{\sigma_u = \sigma_v\}} | \hat{\sigma}_u = \hat{\sigma}_v = +)^2 \\ &\leq \frac{1}{4} - \frac{\delta^4}{64}, \end{aligned}$$

hereby indeed violating (4) for large n on an event that has probability $\frac{n/2}{n} \cdot \frac{n/2-1}{n-1} \rightarrow \frac{1}{4}$, for large n . \square

We summarize these results in the main theorem of this paper:

Proof of 1.2. Combine Theorem 4.2 and Lemma 4.3. \square

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